FUZZY JUDGMENT AGGREGATION: DISTANCE BASED METHODS

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Submitted to the Lahore School of Economics
in partial fulfillment of the requirements for the degree of
MPhil Economics

Date: 2016

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Abstract

Arrow (1963) established that a group cannot always reach logically consistent collective outcome. Subsequently many developments like premise based, conclusion based and distance based methods have emerged in literature to reach group consistency in crisp logic. This study is focused on the judgment aggregation in fuzzy logic based setting with novel involvement of family of t-norms. We compare four distance based methods due to Miller and Osherson (2009) using Łukasiewicz and min t-norm. These methods in fuzzy setting gives closer results to consistency of outcome. It also broaden the set of properties and authenticity of the methods. Distance method in our study satisfy Arrow's axioms in solution method that previously failed in crisp logic.

Acknowledgments

First and foremost I would like to thank Almighty Allah who gave me the strength and showed me the right path to my life. After Allah I bestow all my success and achievements to my great mother who was like a constant support, a strength, who trusted me when I lost faith in myself. She stood there to lift my spirit and morale. After her I would thank Syed Ijlal Ahmed, who without any hope of return pushed me to work hard and is one of the most important person in my educational career. My supervisor Dr.Ismat beg, who is the most amazing person I met in my life, not only he was always there to guide me through the entire process but also he is a mentor and an inspirational figure. Most importantly my co-supervisor Dr.Naved Hamid who took me in his wings when I needed the most support. I pay my special regards to Miss Asma Khalid who was there to guide me at the start and till the end. Last but not the least I would like to thank all my friends and family who supported me in this process and tolerated my impatience.

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1. Introduction to Judgment Aggregation

Since 18th century aggregation theories have been of considerable importance for the social scientists with significant research work done. These theories shed light on how individual judgments are collected are merged into judgment and how democratic majority fails to give consistent outcome. This problem arises as one of the most fundamental barrier of collective decision making in various situations. Ranging from the most basic collective decision making parliaments to small one case based committees, from economic expert panel to court judges' committee. Judgment aggregation is also explained as a tool for interdisciplinary problems that they have in common.

The latest attention to judgment aggregation was triggered by the so called "Doctrinal Paradox" or "Discursive Dilemma" and the name was given by Kornhauser (1992). This problem was primarily faced when the situation was multifaceted. In literature it was first discussed by Kornhauser and Sager (1896), where they explicated a three member court. The court had to arrive at a judgment in a breach of contract case; comprising of three propositions which have to be true in order for the defendant to be held liable. By propositions we mean a statement on which decision is to be made. Here propositions are denoted by "a", "b" and 'c'. Where 'a' states that the defendant was not liable to do an act, 'b' states that the defendant did that act, and c the conclusion premise stating a person is liable to pay compensation. Judges are then to state if premises are true or false called judgment, together the decision on all the premises written individual or collective is called as judgment set.

Important to know is that 'c' is true if both "a" and"b" are true, which means that a person was liable not to do an act and still did it. Hence he is liable to pay compensation. Together $(a \wedge b)$ holds a person guilty of breach of contract. The following table portrays the Doctrinal paradox.

	a	b	c = if both a and b
Judge P	True	False	False
Judge Q	True	True	True
Judge R	False	True	False
Majority	True	True	True/False

Table 1: Doctrinal Paradox

Analyzing Table 1, we can see that first judge holds first premise to be true but not the second one. Second judge find both the premises to be true, however the third judge apprehends first to be false but second to be true. Majority voting holds two premises to be true but the collection of third turns out to be false, resulting in judgment set $\{TTF\}$. Likewise Table 1 majority outcome means that the person was liable not to do an act and did that act but is not guilty of the breach of contract.

Pettit (2003) explains this paradox as discursive dilemma'. But the question was how general is this doctrinal paradox if it is limited to some specific case of majority outcomes or it's not possible for any majority to reach a consistent outcome. In response to this, List and Pettit (2002:2004), presented a model of judgment aggregation, which was inspired by none other than

Arrowian social choice theory in Arrow (1963) and illustrated a generic form proving the impossibility theorem and inability of the majority outcome to reach a consistent judgment.

After List and Pettit (2002), first discussed model of judgment aggregation, much debate started assessing if judgment aggregation is familiar with the preference aggregation. Preference aggregation is endorsed with problem of cyclical preferences, judgment aggregation similarly bear the consequences of inconsistent majority output. However preference theory is a generalized form of judgment aggregation. Likewise the concept of “discursive dilemma” resembles Condorcet paradox. Dietrich and List (2007) explained the analogue of Arrow’s theorem in theory of judgment aggregation. Arrow (1963) started working in the field of social welfare functions and analyzed how many of them satisfy specific axioms. Arrow proved that there is not any method in preference aggregation that satisfies five credible axioms. This deductive analysis of results was named as “Arrow’s impossibility theorem”. This theorem provoked many debates and research in social science fields. Riker (1982) interpreted Arrow’s theorem as one of his mathematical proof of impossibility of populist democracy.

2. Literature Review

This section gives an initial survey of the hypothesis of judgment aggregation. It presents the idea of voting that initially roused the field, clarifies a few key results on the outlandish possibility of proposition wise judgment aggregation, introduces a pedagogical confirmation of one of those outcomes, examines relaxation from the inconceivable possibility and relates judgment aggregation to some other striking aggregation issues. Illustrative instead of comprehensive audit is expected to give more ideas to those who are new to the field of judgment aggregation a sense of this quickly developing exploration territory.

Benamara et al. (2010) suggested that in numerous choice issues the conclusion is more important than the reasons behind it. As per them, while procuring an applicant case in point, one is more worried of which new person is to join the office than of the purposes behind picking her. They also suggest that considering just the individual judgments on the conclusions has also the point of interest like premise based system. Be that as it may, then, in popularity based societies, individuals have the privilege to question the process of the choice making. Along the same lines Dietrich and List (2008) worked on the aggregation of probabilities in preference ranking, modelling belief of agents between a range [0, 1] to rank the alternatives. This relates in spirit and structure with judgment aggregation in fuzzy setting, when individual judgments in terms of degree of membership are aggregated to form a collective outcome.

2.1 Economic Rationale and Linkages

Collaboration between multiple agents is based on cooperation and coordination. There have to be group decisions which should be reached by the agents as superior over other factors like their own opinions and needs. A kind of group assent is called agreement. An agreement is mutual and binding cooperation among agents. But the whole process of reaching the agreement has become as a major research field in economics, considering cases where a decision has a social impact.

Social choice theory in economics has worked in order to reach method that helps individuals to reach a decision that is chosen from a given set of alternatives. In the horizon of social choice theory there are different theories, preference theory, voting theory, utility theory and judgment aggregation theory. They all have focused on shaping formulas and procedures to reach coherent collective outcomes.

Preference aggregation deals with the problem of formulating a group decision on a given set of alternatives. In this each agent state out its most preferred alternative followed by what he prefers less, giving a set of preference order. Preference aggregation modeling then deals with finding an order that is representative of the whole group.

Along the same lines we have voting theory, in this theory we select a winner through majority decision from a number of candidates. Each agent cast its opinion in form of a vote and then selects one candidate over the other(s). There are different structures on which voting is based, ranging from one-to-one to multiple candidates, this is basically a form of preference aggregation. There are many recognized situations where voting is an integral method to find the best contestant. These setting involve voting for the establishment of democratic government as well as for a small event like selecting the best player. Voting rule is based on individual selection of agents of one candidate over the other(s).

Utility theory has been described in economics that it is not possible to measure the satisfaction an individual derive from consuming a good. However it is possible to rank alternatives in order of preference of the consumer. Choice of alternatives is constrained by income and prices hence a rational consumers spends on an additional good only if its marginal utility is same or greater than the last unit. Utility has been used by economist to draw indifference curves, these curves plot the combination of commodities that an individual or society deem acceptable for a set level of satisfaction. These utilities are then used to construct social welfare functions. When joined with production and constraints these functions are used to analyze Pareto efficiency. Pareto efficiency ensures that the outcome picked up should give maximum aggregate social welfare. Utilities have also been a fundamental method to derive revealed preferences where direct observation was not possible, thus economist would devise a way to infer utilities from observed choices e.g., in people's willingness to pay. Utility theory is important as it shares resemblance with judgment aggregation in fuzzy setting. Here we are trying to formulate a collective solution that is close to individual judgments and maximize social welfare. Plous (1993) draws attention to the Bayesian belief network, and situations when information is uncertain and conclusions are drawn based on probabilistic inferences. Pearl (1998) suggests that in uncertain, complex and unmanageable decision problems Boolean networks are useful. Gumasta et al. (2011) developed an index to aid decision makers in manufacturing systems with the help of multi-valued utility theory. Soufiani et al. (2012) worked on random utility theory for social choice, they model preferences of agents through a real valued score and then ranked them according to it. This approach is widely used to analyze political elections.

Judgment aggregation on the other hand deals with the issue of making collective decision concerning truth-values of multiple issues. Each agent sets out its judgment in form of true or false about truthfulness of each issue, in crisp logic it has been a binary value representing whether a concern is true or false. A judgment aggregation rule signifies the form and combination through which the truth-value assignments are combined together to reach as collective solution. The theory basically questions how the numerous judgments on logically connected propositions can be aggregated into consistent group outcome.

This idea from preference aggregation can be extended to judgment aggregation. Preference aggregation with its roots in utility theory has also involvement in theory of judgment aggregation. Dietrich and List (2006) constructed an embedding of preference aggregation into judgment aggregation. They stated that preference aggregation problems can be modelled as special cases of judgment aggregation. They presented preference ordering as sets of binary ranking judgments in predicate logic. List and Pettit (2004) derived impossibility result on preference aggregation from their earlier results on judgment aggregation. List and Polak (2010), further explained how judgment aggregation can be seen in new setup with different impossibility results and consequently finding escape routes.

2.2 Arrow's Impossibility Theorem in Light of Condorcet's Paradox

After Arrow (1963) classic book "Social Choice and Individual Values", the theory of aggregation has become a booming area for research. Arrow's book emphasized on the aggregation of preferences, which we previously explained as a set of ordering by a group. Since there are many economic and political situations where preference ordering is required, and thus the work inspired many researchers. But the link between social scientist and this field goes back to the Condorcet, where the argument which voting method to use is still in debate.

Condorcet (1785) explained his analysis by an example where individuals had to vote on three candidate.

Voter	**A is preferred over B**	**B is preferred over C**	**C is preferred over A**
1	Y	Y	N
2	N	Y	Y
3	Y	N	Y
Majority	Y	Y	Y

Table 2: Source: List, B. Polak, Journal of Economic Theory 145 (2010), 441–466

In Table 2, if we apply transitivity property we are able to distinguish there are various orders of majority preferences, resulting in a questionable choice of winning candidate. The cyclical preferences are represented as $A > B > C$. Following this McGarvey (1953) generalized this work further by evolving a method for voting procedures in case of finite candidates. Stearns (1959) uses the previous work to develop method that require fewer voters to produce the anticipated outcomes. However voters can manipulate their preferences in order to achieve this desired outcome explained by Riker (1982), which was further affirmed by Levmore (1990) who suggested the involvement of activity level of political party and the election consistency. However recent work by Tullock (2000) counter argued over the fact and said that none of the studies considered real life examples of voting, he suggested that there are methods employed by government to overcome any cyclical preferences.

Now what we call preferences can be interpreted in various democratic situations in different ways. There have been numerous economist to shed light on how these preferences can be achieved and described. From Condorcet to John Stuart Mill to Adam Smith and Immanuel Kant. However for explaining purposes we shall consider all the different forms to be described in one broad category of ‘preferences’.

In Arrow’s social theory there has been a central value which was democracy, as each member of the decision making process should have equal representation. The entire discipline and research have kept this as their very basic rule to be followed. For example, Sen (2012) explains this as major embarrassment for a set of axioms if the structure renders dictatorship existence. Arrow impossibility result on the other hand have a far reaching impact on scientist from all fields like social science, economics and psychology. However an inside look explain how the implications often attached to this theorem might be misleading. Arrow was concerned with finding an output that is the aggregate preference ranking over potentially available social states, this ranking is based on the collection of the individual preferences over these states and named this as social welfare function.

Sen (2012) explains Arrow’s original axioms when taken together results for the impossibility theorem. Unrestricted domain, which states that for any set of individual preferences the ranking is reflexive, transitive and complete. Independence from irrelevant alternatives which means that any social ranking for pair x and y depends only on individual ranking of x and y.

Pareto principle, this axiom states that if each individual prefer x toy then majority should also prefer x to y. Non-dictatorship ,this axioms means that each individual is treated equally and no one prefers x on y can force any other individual to follow. Now the impossibility arises is when if there is any three distinct social states and set of individuals, then there is no function that satisfy all four axioms. However all these axioms were so strong and popular that they could not be rejected either

In order to avoid the negative conclusion of Arrow's theorem, experts worked out various ways to overcome the difficulty in making choice. There have been two main branches which have focused after Arrow's theorem

1) research of functions with domains comprising of profiles of individuals
2) research of other aggregation rules

Under the heading of preference profiles there have been numerous work in finding aggregation rules. Kirman and Sondermann (1972) suggested that if the assumption that there are infinite individuals is replaced by finite then we are able to find aggregation rules that follow all other Arrow's condition. But these rules are of less use as these involve complex mathematical calculation, they also further explained that there are invisible dictator behind such aggregation functions. There are some mathematical restriction on such rules. Mihara (1997, 1999) worked to exhibit how these rules violates algorithmic computability.

Number of alternatives in such problems amounted to great deal of interest in itself. May (1952) theorem explained majority credentials said that there have been certain fulfillment of axioms if simple majority is followed. But Arrow pointed out the problem in case of three or more alternatives which does not give a robust majority output. Following this and May's interpretation, Nakamura (1978) laid out a more general solution of the problem stating a number named

‘Nakamura number’ that satisfy the axiom if below a certain number and does not satisfy axioms if above this number. Thus a common way around Arrow’s axiom is that we use majority output as long as there are two alternatives. But if there are more alternatives then we can pair them. But this pairing also causes the problem of Pareto efficiency, as the order by which alternatives are paired have an immense impact on the winning capability of the alternatives. All this process when viewed as a game we see Arrow’s theorem still holds true.

Arrow’s axiom on domain invited considerable criticism from the scientists, his approach was that domain should be universal. To work around this condition, by restricting the domain.

The independence of irrelevant alternatives condition has seen the most literature among all other axioms. Employing independence in aggregation means we proceed by aggregating opinions about each proposition separately from the others. This approach however has very important, as this is a strict rule to form a consistent collective outcome.

The impossibility conclusion of the Arrow’s theorem could be deduced from independence rather than systematicity, through this we can better understand the problem of discursive dilemma. This way we could also link it with the social choice theory where the strongest assumption is that of independence of irrelevant axioms.

The basic approach to look in this direction was started by Dietrich (2006) and Pauly and van Hees (2006) who hypothesized independence as the starting point, stating solution rule has to satisfy both non-dictatorship and independence axiom. Dietrich (2006) crack deeper onto this idea and stated that the results hold true only in case of solutions restricted to particular set of axioms is a constant function. Mongin (2008) went ahead and explored the subsets of agenda and concluded that there is no solution method that satisfies all axioms together.

Following these studies the problem of Condorcet paradox and Arrow's impossibility theorem in judgment aggregation was generalized. Kornhauser and Sager (1986, 1993) through an example of jurisprudence lit the spark in the field of judgment aggregation. List (2001) explained the problem of inconsistent majority output and discursive dilemma. List and Pettit (2002) presented the first model of judgment aggregation, where they combined all the aggregation rule with Arrow's axiom and logical presentation of propositions. The impossibility theorem in judgment aggregation directs us in two directions, one we know that there is no ideal rule for aggregating judgments but also that in research we should find alternative methods by relaxing these conditions.

With the help of this model they proved impossibility theorem, stating no judgment rule can satisfy Arrow's axioms. Working on the same lines Pauly and Van Hees (2006), in their study on independence axiom, say that let ʠ be classical language for proposition closed under negation, the solution rule thus has to satisfy both non-dictatorship and independence axiom. Nehring and Puppe (2010) gave an idea about oligarchs and defaults. Oligarchs are the normal people and defaults are the dictators. If the former agree over a formula and on collective judgments the results are not unanimous then defaults chooses between the two alternatives of judgment sets. The solutions due to Pauly and Van Hees (2006) and Nehring and Puppe (2010) did not fulfill Arrow's independence axiom. In their work they relaxed the axiom of independence and dictatorship.

2.3 Aggregation Operators

There have been much focus on the rules that were exercised in voting theory. These rules established certain criteria as to how the winners were picked. More importantly how manipulation might be controlled and the necessary information needed to reach a reliable outcome and etc. Few research have defined judgment aggregation rules. Starting from the ‘premise based rule’, which was basically clause by clause voting explained by Dietrich and Mongin (2010), here each judge state its judgment on each clause of the agenda. For this method , the agenda is partitioned in two direction, one is premises on which the judges give their opinion and then the conclusion is drawn based on the judgments on the premises either through logical channel or some other constraint defined before.

However in the conclusion based procedure, individuals can decide on premises on their own in private and then just state their judgment on the conclusions. Other methods involve sequential procedures, these methods is when the agenda components are considered in a set order, and the decision on the prior premises affects or limits their decision on the latter one.

Moreover, the decisions are based on the choice or order by which the path of the premises is determined explained by Dietrich and List (2007) and Li (2010). Following their own work Dietrich and List (2007) also explained the working of quota based rules, here each proposition is linked with a quota. The premises are accepted only if the number of judges accepting the premises is above the defined quota. Dietrich and List (2007) also explained sequential quotas along with uniform rule which gives same number of quota for all the elements of the premises. Among others distance based methods have seen immense importance in the field of judgment aggregation and link to the theory of belief merging.

2.4 Distance Minimizing Methods

The theory of judgment aggregation has been attached to other theories like abstract aggregation by Wilson (1975) and belief merging by Konieczny and Pino Perez (2002) and Pigozzi

(2006). Pigozzi (2006) expressed the idea of belief merging in judgment aggregation and has contributed to this field. Belief merging inherits the problems and limitations of doctrinal paradox from voting theory. Each individual states his/her own belief on a proposition, this is then taken to be mapped on to a number 1 for true and 0 for false, $P \rightarrow \{0, 1\}$.

We will be looking at the literature of belief merging to understand the problems this approach faced. In this approach distance metrics were involved.

"Instead of considering each proposition one by one, belief aggregation employs distance metrics to arrive at collective judgment set at one " (konieczny and Pino Perez 2002).

There is another set of literature that talks about the methods of distance based judgment aggregation. Distance methods have been one of the most relied upon methods in the field of judgment aggregation to find solution for multi criteria multi person situations. Pigozzi (2006) explained the merging of belief bases in order to arrive at a decision that does not fall in discursive dilemma trap. Idea of distance based approach was to minimize the distance between judgment sets and arrive at a conclusion that is a better compromise between differences of opinion**.**

Hamming distance is a method to find distance between two judgment sets. Pigozzi (2006) defines Hamming distance as the "distance between judgments sets as the number of propositions on which they disagree".

Hamming metric is the most extensively used metric in literature. Explaining Hamming we start by supposing two judgment sets**,** ¬ means negation of the proposition.

$$p = \{x, y, x \wedge y\}$$

$$p' = \{x, \neg y, \neg (x \wedge y)\}$$

distance between them is 2, meaning the difference of opinion between p and p' is over the proposition x and$\neg\,(x \wedge y)$. This idea is observed in a different way by Duddy and Piggons (2011) where they say that the disagreement over $\neg(x \wedge y)$ is the result of disagreement over y and hence Hamming distance might be double counting.

Duddy and Piggons (2011) came about another calculation of distance between two judgment sets, they explained the idea through the vertex and then the distance between the above two judgment set equals one even though they disagree on two propositions. Miller and Osherson (2009) built on the idea of Hamming distance and came about different methods in Hamming distance based approach to reach a set of consistent judgment. For every solution selection of a different judgment set yields a different solution rule. Discussion of these methods will be considered in section of methodology.

2.5 Majority Preserving and Distance Minimization Methods

2.5.1 Rules Based on Majoritarian Judgment Set

Let an aggregation rule be defined by alphabet “G”. A rule G is said to be based on the majoritarian judgment if there is a function that has a solution rule that is the majority judgment set (M(P))= solution. If there are two set of profiles and they have individual sets that yield the same majority output then the solution they give will also be same. Here we imply that the majority set is consistent and will be our solution. However situations where majority does not give a consistent outcome, there are then methods to get rid of the inconsistency either by eliminating some agenda items.

2.5.2 Rules Based on the Weighted Majoritarian Judgment Set

In methods based on majority outcome only the information regarding the majority output is saved, stating that these propositions got the majority votes. However in weighted average the information is much more than majority. It takes into account the number of voters that voted in favor of a proposition. Rules of this family are sensitive to the number of agents who supported a proposition. Majority output can be deduced from weighted average but we cannot find other way around. This can be viewed as the counter method of voting rules based on weighted majority.

2.5.3 Rules Based on Distance

There are two types of distance based rules, one is where we are looking for minimizing the distance between judgment sets without changing the actual profile and thus finding solution this way. However there are other methods that work on making the distance minimum and then making relative changes which are minimal to the profile. The whole process of this judgment set minimization of distance is to actually guarantee some sort of compromise from the least satisfied individual. As by taking distance we will not be looking at the majority outcome but also the judgment set of each individual and hence taking into consideration as to how much each judge feels about the truthfulness of the proposition.

2.5.4 Rules Based on Removal or Change of Individual Judgment

This rule looks for a customized profile that has the minimum distance from the majority profile. The whole process is done to achieve majority consistent profile. Here by applying distance method to arrive at the closest adjusted profile and for every distance method the profile achieve is different. This method offers much more than it looks from outside including the personal beliefs of the individual and we can see some considerable improvement when we apply fuzzy set theory in this method.

3. Methodology

In this section, we will state the methodology that fuzzy set theory and judgment aggregation has developed in existing literature, as well as how the amalgamation of both the fields works in our research.

3.1 General Framework of Fuzzy Logic

In prevailing aggregation process we found that there is no process that allows the consistency of group outcomes and yet fulfills all axioms in crisp logic. Thus there has been much work done in the field to approach a consistent outcomes using fuzzy logic.

In utilizing ordinary characteristic dialect to bestow learning and data, there is a lot of imprecision or fuzziness. Such explanations as "Amna is tall" and "Ahmed is youthful", "power is great", "device is efficient" are not strongly characterized. To draw induction from them we require some instrument which is less strict.

On the off chance that the conclusion of a voter is something between a strict YES or NO then he ought to have the opportunity and flexibility to express it, else, a general public can't think of an aggregate choice which best speaks to the decisions of its people. To record for this, we may utilize multivalued rationale or fuzzy rationale. Basic fuzzy rationale with just truth utilitarian connectives can be gotten by "fuzzifying" established crisp proposition. Fuzzy set hypothesis was presented by Lotfi Zadeh in 1965.

"A fuzzy subset of a non-empty universe U is a mapping $U \longrightarrow [0,1]$"

Then we can assign any value between the number 0 and 1, this is called the membership function, and the number assigned is the degree of membership in set A of u.

To explain the idea, consider the universe of “Pakistani citizens”, let A be the fuzzy subset of “young citizens”. Then we can assign any value between the number 0 and 1, this is called the membership function, and the number assigned is the degree of membership in set A of u.

To explain the idea let X be the set of “Pakistani citizens”, and

A be the fuzzy subset of “young citizens”.

The membership function of A is defined as follows:

$$A(x) = \begin{Bmatrix} \frac{1}{20}x & if \quad x \leq 20 \\ 1 & if\ 20 \leq x \leq 40 \\ \frac{60 - \mathrm{x}}{40} & \text{if } 40 \leq \mathrm{x} \leq 60 \\ 0 & \text{if } 60 < x \end{Bmatrix}$$

If a young person is to perceive it, they might consider anyone under the age of 25 to be young, however a male in his fifty's thinks otherwise. Hence there is not a clear truthfulness of the statement that is universal. For this case we define a function where if age of any person is between the range 20 to 40, degree of membership of 1 is assigned, meaning he belong to this group completely. But in the age bracket from 40 to 60 there is a function that decides the truthfulness of the statement based on your age. However a zero degree of membership is allotted to someone who is not below 60 rendering them not to be young.

Expressing the graph in Table 3, we may see that the graph is not smooth rather made up of certain segments. From $0 \leq x \leq 20$ an upward sloping line, followed by a straight line in the range $20 \leq x \leq 40$. Downward sloping line in the range between $40 \leq x \leq 60$. This then suddenly touches the y=0 line when x becomes greater than 60.This graph is the representation of fuzzy theory in numbers where there is no clear black and white rather shades of grey which occupy most of the problems in our real life.

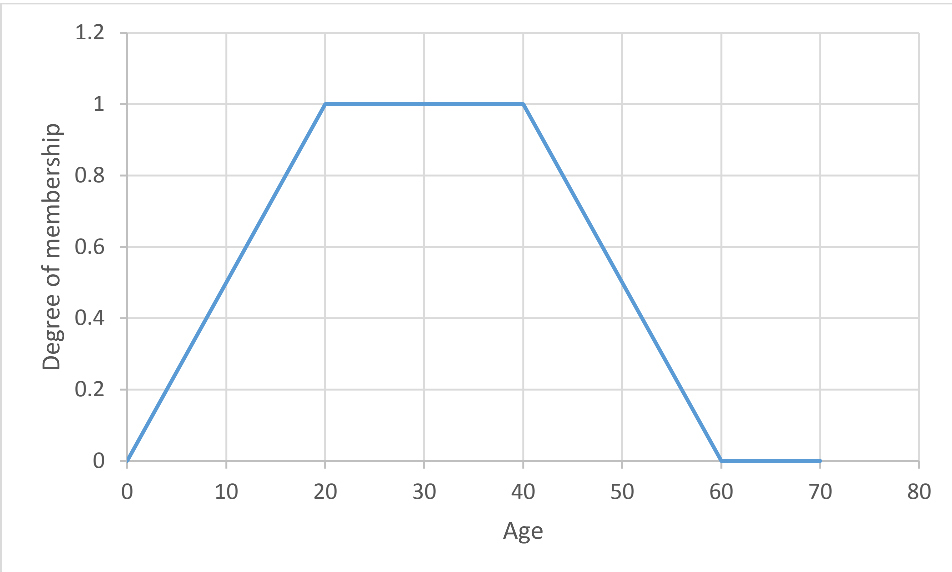

Table 3: Membership function A

The theory of fuzzy logic is filled with numerous aggregation operators known as triangular norms and their implications. These aggregators have played a substantial role in forming some of the important insights. We will discuss few of these aggregators in deep as these operators will be one of the core ideas in our methodology.

The degree of membership in our paper represents the general belief of the judge. Pauly and Van Hees (2006) generalized paradox using multivalued logic, also claiming that this does not ensure collective rationality. Duddy and Piggons (2011) also presented the judgment aggregation model where judgments were in degrees rather than the two-valued crisp numbers. Dietrich and

List (2007) worked on the development of propositional attitude aggregation. Non binary beliefs were also explained by probabilistic approach.

Beg and Butt (2012) speaks about how judgment aggregation can be converted to other than a true and false interpretation and then coming to a feasible conclusion. Beg and Khalid (2012) also draws attention to the use of other variables than just true or false in order to formulate aggregation in preference modeling. However here I am trying to model the distance based approach; hence it will be of our interest to look into the methods initially used to convert classical logic into fuzzy set theory. In the paper of Beg and Butt (2012), they tried modeling judgment aggregation in fuzzy set theory. Pigozzi (2009) also used the distance method approach in classical setting and converted it into fuzzy framework

All aggregation calculation is based on the idea of simple majority. Various literature emphasize on this notion and its importance with regard to credibility of the solution. Smith and Banks (1996) establishes idea on majority voting as more credible, based on the assumption that individuals are more sincere when they are making group decisions. Sincere behavior is not always backed by rational choices and do not form a Nash equilibrium. Surowiecki (2004) debate the same notion and emphasized that pooled decisions lead to a better set of judgment over an objective and maximize the results credibility. In this regard Goldman (2004) and List (2005) stated the importance of both correspondence as well as coherence in decision making process.

Hence with the help of distance methods based on majority outcome a complete rule can be defined as the solution rule. Miller and Osherson (2009) stated this method would generate more consistent collective judgments. They further built on the idea of Hamming distance and executed different methods in Hamming distance approach to reach a set of consistent judgment set. For every solution the selection of a different judgment set yields a different solution rule.

3.2 Model of Judgment Aggregation

This study has used Dietrich's (2007) model of judgment aggregation in general logic and construct a model based on fuzzy logic in judgment aggregation derived from Beg and Butt (2012).We will start by explaining a set of people, required to make decisions on propositions with logical connectives, these propositions are closed under negations.

Explaining some basics of our model, let £ be a logical language (called propositions previously) with connectives. An agenda is a non-empty set $X \subseteq £$, interpreted as a set of propositions on which judgments are made, with X as a union of proposition-negation pairs. Now given our agenda, each individual i's judgment set is an element p_i.

Let X be an agenda, $N = \{1 \dots, n\}$ where $n = 5$, and a profile is a set P=$\{p_1, \dots, p_n\}$ where each p_i is a judgment set.

It is important to know that the judgment set has to be complete as well as consistent. By complete we mean the individuals decide on all the propositions and by consistent we mean they follow the rules of the general truth table. In our model the profile contains all the judgment sets. Utilizing profile draw and distance based formula from Miller and Osherson (2009) explained in Table 4.

Judges	**x**	**y**	$x \wedge y$
e_1	T	T	T
e_2	T	T	T
e_3	T	F	F
e_4	T	F	F
e_5	F	T	F
Majority	T	T	F

Table 4: source: Miller, M., and Osherson, D. (2009). Methods for distance-based judgment aggregation. Social Choice and Welfare, 32(4), 575-601

3.3 Generalization of Distance Based Methods in Fuzzy Logic:

However before explaining our distance formula, we will explain the basic Hamming distance represented by "H". Hamming distance between two sets of judgement on given proposition, is the number of propositions on which they disagree. If there are two judgement sets p_i *and* p'_i, then Hamming distance is $H(p_i, p'_i)$ Pigozzi (2006) .

If p_i is {TFF}={1,0,0} and p'_i {FFF}={0,0,0}, the Hamming distance becomes as

$$H(p_i, p'_i) = |1-0| + |0-0| + |0-0| = 1$$

We use the distance formula by Miller and Osherson (2009) as

$$d(p_i, p'_i) = \sqrt{H(p_i, p'_i)}$$

There are two ways Hamming distance work, either we minimize Hamming distance to profile or we minimize Hamming distance to majority outcome from established crisp sets in this way we approach both consistency and closeness to individual judgements.

This reasearch uses Hamming distance as a technique to find the minimum distance between “s” established crisp judgement set and individual judgment sets. Here “s” the established crisp judgment set are referred as interpretation. Pigozzi (2006) describe interpretations are model of propositional formula that make the statement true. For example if we formulate a problem where judges have to decide to give scholarship to a Ph.D student based on the following criteria:

a: $Experience$

b: $Number\ of\ publications$

c: $Scholarship$

The committe decides to give scholarship if only if the candidate possess both the experience as well as publications. For this we look at all possible combinations that makes the solution statement true, our interpretation set are:

$$s = \{\{TTT\},\{TFF\},\{FTF\},\{FFF\}\} = \{\{111\},\{100\},\{010\},\{000\}\}.$$

Where $s_1 = \{111\}, s_2 = \{100\}, s_3 = \{010\}\ and\ s_4 = \{000\}$

We take distance of all of the interpretations from any profile “ p” and find the minimum distance and that is our solution rule.

Using the four distance methods explain in the section 3 and 4, are applied in fuzzy setting to see if our methodology satisfies the research aims. Methods have been based keeping in view majoritarian rule in order to appreciate the basics of democratic credentials, this was justified by

May (1952). Four methods that we will be using in this study are 1) Prototype 2) Endpoint 3) Full and 4) Output.

We start by giving a brief overview of each of the distance methods used in this study. Prototype is the sum of distances across all the individual judgment set in profile p from a consistent judgment set. The one consistent judgment set that has the minimum distance from the sum of individual judgment sets will be the solution rule. Prototype satisfies supermajority responsiveness for all three classes of metrics. However Prototype satisfies preservation and sensibility for classes of metrics that are separable only Miller and Osherson (2009).

Endpoint on the other hand is the most straightforward method among all the others; it takes into account just the consistent judgment set that is the outcome of proposition wise majority voting. Endpoint is easily applicable as it looks at only the aggregate outcome of the decision, but it's insufficient as it does not involve individual judgment set and yields same results even if the individuals have different sets of judgment. Endpoint fail Supermajority responsiveness for all sets of metrics, but fulfills preservation as well as sensibility for all sets of metrics.

Full and output are methods that involve the procedure of judgment revision. These two have not been much discussed in literature, however considerable work has been done on methods requiring the revision of judgment until we arrive at a consistent judgment set.

Output is a unique method that took into consideration the closest profile of judgment sets (individually consistent or not) that yields a consistent majority outcome. However Full looks at the closest consistent profile to reach a consistent majority output [Miller and Osherson (2009)].

This study models the distance based approach in fuzzy logic. It will be of interest to look into these methods initially used in classical logic. In the paper of Beg and Butt (2012), they modeled judgment aggregation in fuzzy set theory. They also used the distance method approach in classical setting and converted it into fuzzy framework.

In this thesis, multivalued rationale or fuzzy rationale will be utilized to achieve desired results, by means of t-norm operator in order to arrive at our methodology. Going into detail, I will start by explaining t-norm and how it will be used in my setup. Beg and Samina (2008) explained t-norm or triangular norm as functions with properties increasing associative and commutative. They are described as binary operations on a Cartesian product in a way that they map it onto a range of real number between 0 and 1.

Triangular norm (t-norm, Δ) is a function

$$\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1] ,$$

Satisfying

1) $\Delta(x, 1) = x.$ (Identity element)
2) $\Delta(x, y) = \Delta(y, x)$ (commutative)
3) $\Delta(x, (\Delta(y, z)) = \Delta\, (\Delta(x, y), z)$ (associative)
4) $\Delta(x, y) \leq \Delta(x, z)\, if\ y \leq z$ (monotonicity)

for all x, y, z ϵ [0, 1].

In our study we will use Łukasiewicz t-norm denoted by Δ_L

$$\Delta_L(x, y) = max\, \{0, x + y - 1\} ,$$

to arrive at an improved solution method. Beg and Butt (2012) explained use of Łukasiewicz t-norm in distance based judgment aggregation. However still we need to find out if this methodology brings about better results.

It will be worthwhile to explain the intuition behind the use of Łukasiewicz t-norm in our thesis. Important reason concerned the use in the previous work. Łukasiewicz t-norm has been frequently used in literature due to its similarity with the mean aggregator. As our range for degree of membership lies between 0 and 1, Łukasiewicz t-norm works to find the average, a strict average though but that makes our results more reliable. Hence to avoid extreme values we have chosen this t-norm. It is important to highlight that other t-norm like Min and Max are likely to give biased results when values are either 0 or 1. They are likely to pick the extreme values. Extreme values are not the best representative of the opinion of the judges.

3.4 Defining Classes of Metrics:

There are three classes of metrics that we will be considering, a distance metric d is normal if it satisfy a non-decreasing property as disagreement grows. Miller and Osherson (2009) assigns a function that maps the set of propositions on which the two sets disagree onto a real number. The two sub-classes of normal metrics are increasing and separable. By increasing it implies that the distance metric d is increasing for all non-empty disjoint sets.

Also a metric is separable if it contains dense countable objects. These classes of metrics are also defined by the characteristics of the solution rule that is used while aggregating judgments and what we will be considering in this paper.

There are three basic characteristics, supermajority responsiveness, which means that while applying an approach to reach a solution there should be some consideration given to individual level judgments. Second important characteristic is Preservation, which requires that a solution rule should recognize majority rule as the output if it is consistent.

Lastly sensibility provides a set of criteria for a judgment set to be considered as solution rule. One is that the judgment set is the closest consistent set that minimizes the distance from the majority output. It differs from majority output only on one proposition and the proposition that is being altered has the minimum majority margin among all the other propositions.

4. Distance Based Methods in Fuzzy Set Theory:

In this section, we formulate mathematical formulas and their calculations in order to analyze how judgment aggregation works in fuzzy setting. The four methods considered include Prototype, Endpoint, Output and Full.

4.1 Prototype

Prototype in crisp logic is an aggregate distance of each judgment set from the consistent judgment set. Judgment set that has the minimum distance from the aggregate of distances is deduced as the solution rule. The four established consistent judgment sets are $\{TTT\}, \{TFF\}, \{FTF\}\ and\ \{FFF\}$.

Taking the distance of defined logic set from each of the judgment set one by one and aggregating. Basically finding the total distance the established crisp set has from all the individual judgment set. Measuring the distance we compare how the distances are from other crisp sets, the one set that give minimum distance becomes the solution for our approach.

We define this method with an illustration of an example. We allowed judges to express their opinion in fuzzy setting. Which means instead of a TRUE or FALSE as an answer they express their opinion and assign a degree of membership to each of the proposition on a range of numbers from [0,1].

Three propositions set, each judge has to give his judgment on these propositions based on their knowledge and information. We take an example of a committee who has to decide on the emission of carbon dioxide in an area that has civilian settlement around it. Decisions are made on the stated propositions, but each judge has a complete rationale behind the process based on their knowledge and experience.

The following propositions were part of the agenda

a: $Carbon\ dioxide\ gas\ is\ dangerous\ for\ health$

b: $Carbon\ dioxide\ emissions\ are\ effecting\ daily\ life\ of\ the\ local\ residents$

c: $The\ emission\ of\ carbon\ dioxide\ should\ be\ reduced$

Proposition “a” and “b” are based on the degree of membership or truthfulness that judges assign to each proposition. Proposition “c” is our conclusion clause and will be calculated based on the judgments on premises.

Hence before conclusion clause was calculated by crisp aggregators. Additional to literature we propose and apply fuzzy logic operator to calculate the conclusion premise. Analyzing the profile draw in Table 5, there are three proposition and each judgment set comprises of degree of memberships assigned to each of the proposition by the judges, “$a\ \Delta_l\ b$” is determined by Łukasiewicz t-norm which is explained by

$$\Delta_l(a, b) = max\{0, a + b - 1\}$$

Before calculation, we justify our selection of the t-norm and purpose behind deciding on it. There are many fuzzy logic operators but in this example we will be using Łukasiewicz t-norm. Reason for using is that it's a strict t-norm and does not give a verdict unless there is high degree of membership for each proposition.

The example above involves lives of the people as well as the production and commercial profits are at stake and there should be really strong belief about the truthfulness of the proposition before the decision is rendered in favor of restricting the emission of the factory.

Judge	a	b	$a\Delta_l b = max\{0, a+b-1\}$
1	0.40	0.70	0.10
2	0.30	0.80	0.10
3	0.90	0.30	0.20
4	0.70	0.40	0.10
5	0.20	0.30	0
6	0.80	0.40	0.20
7	0.70	0.50	0.20
$\theta = collective\ outcome$	0.57	0.48	0.13

Table 5: a profile draw; Łukasiewicz t-norm

After we got the data set for "a" and "b", $a\Delta_l b$ is calculated with the help of Łukasiewicz t-norm. Explaining the judgment set in fuzzy setting I will take the judgments of judge 1, $\{0.40, 0.70, 0.10\}$, number 0.40 shows that as per judge 1 the truthfulness to the fact that carbon dioxide is dangerous for health is 0.40. It's a number closer to zero, implying a weaker agreement on the proposition. 0.70 on the other hand shows strong truthfulness of the proposition "b", on conclusion we see 0.10 showing that this judge is not in favor of the conclusion.

After applying t-norm to calculate the conclusion, distance formula is applied to calculate our distance

$$d(p, p') = \sqrt{Ham(p, p').}$$

We calculate the sum of distances of each judgment set from the four established crisp judgment set. The distance were found using the formula:

$$d(s_1, p_1) = \sqrt{|s_{1a} - p_{1a}| + |s_{1b} - p_{1b}| + |s_{1c} - p_{1c}|}$$

Where $s = \{s_1, s_2, s_3, s_4\} = \{\{111\}, \{100\}, \{010\}, \{000\}\}$

Calculating distance from the first established crisp logic set the formula is

$$\sum_{i=1}^{7} d(s_1, p_i) = d(s_1, p_1) + d(s_1, p_2) + d(s_1, p_3) + \cdots + d(s_1, p_7)$$

$$\begin{aligned}\sum_{i=1}^{7} d(s_1, p_i) = {} & \sqrt{|1-0.40| + |1-0.70| + |1-0.10| +} \sqrt{|1-0.30| + |1-0.80| + |1-0.10|} \\ & + \sqrt{|1-0.90| + |1-0.30| + |1-0.20|} + \sqrt{|1-0.70| + |1-0.40| + |1-0.10|} \\ & + \sqrt{|1-0.20| + |1-0.30| + |1-0|} + \sqrt{|1-0.80| + |1-0.40| + |1-0.20|} \\ & + \sqrt{|1-0.70| + |1-0.50| + |1-0.20|} = 8.06\end{aligned}$$

For the detailed calculation on all the sets please see Appendix-I, However when we calculated all the formulas we found the following results.

$\sum_{i=1}^{7} d(s_1, p_i) = 8.06,$

$\sum_{i=1}^{7} d(s_2, p_i) = 7.06,$

$\sum_{i=1}^{7} d(s_3, p_i) = 7.59,$

$\sum_{i=1}^{7} d(s_4, p_i) = 7.54,$

So the minimum distance was from $s_2 = \{100\}$ =7.06. Hence with minimum distance the solution for our method

$$\text{Prototype}_f = \{TFF\}$$

Initially when we looked at the same method in crisp logic in Miller and Osherson (2009) they gave a solution in a form of tie between $\{TTT\}$ and $\{TFF\}$, however using fuzzy setting we have arrived at a single solution and that too is more close to the actual decision as now the distances are less than those in crisp logic. We defuzzify our results to implement conclusion in real life situation.

This can also be applied in economic policy making committees that are based on some specific people having core knowledge in that area. However, this method is of considerable importance as most of the economic decision making is done in form of small committees that comprise of small number of people expert in that area.

4.2 Endpoint

Working on another method, Endpoint, we comprehend that this technique essentially calculated the distance of the consistent judgment sets from majoritarian output; minimum distance between majority output and established consistent judgment sets was the solution.

In fuzzy logic based setting, we take average of the degree of membership on each proposition. Distance from established crisp judgment set is calculated, judgment set with minimum distance becomes solution.

$$d(s_i, \theta) = \sqrt{|s_i - \theta_a| + |s_i - \theta_b| + |s_i - \theta_c|}$$

average of all judgment on proposition $= \theta = \{\theta_a, \theta_b, \theta_c\}$

Now apply endpoint method, on data from Table 5.

Endpoint is a direct method as compared to prototype. After taking group average we calculate distance from the consistent judgment sets as follows. We display the mathematical calculation from "s_1" and remaining is posted in Appendix-II

$$d(s_1, \theta) = \sqrt{|1 - 0.57| + |1 - 0.48| + |1 - 0.12|} = 1.16$$

$d(s_1, \theta) = 1.16$, $d(s_2, \theta) = 1.02$, $d(s_3, \theta) = 1.05$, $d(s_4, \theta) = 1.04$

See Appendix II for detailed calculation. Looking at the results we conclude that the solution with the minimum distance is of judgment set $s_2 = \{100\}$. Hence the solution for our method becomes

$$Endpoint_{fd} = \{TFF\}.$$

The distance from the average when we used fuzzy logic is less than when endpoint is applied in crisp logic. This means that the judgment is closer to all the established judgment set, and the solution we get is a better representation.

As opposed to majority output we are considering every single judgment set and equal weightage is allocated. Hence in calculating θ (majority average), we hold the assumption of independence true by assigning equal representation meaning that every person while calculating group average is assigned equal weightage.

In fuzzy logic based setting we have given equal representation to each of the judge and the solution we get thus is then the better compromise among the judges on the set of proposition.

Previously it was not possible with crisp logic and endpoint failed supermajority responsiveness. Crisp logic might give same results even with different judgment set, as it looks only at the majority. In fuzzy logic, a small change in judgment as much as 0.01 is recorded and reflected in calculations. Hence our method is very sensitive to small changes in the degree of membership.

This method can be applied where there is greater number of the judges, as prototype involves long calculation, however Endpoint avoids it. This is a method that we can use in places like parliaments where the bill has to be passed by the members of the parliament and the number of the members is in hundred. This way the weighted average calculated gives equal representation of each of the member as well as calculation is not tedious and may avoid mathematical errors and gives quick solution.

Two methods prototype and endpoint have few hints which we can verify from further calculations. We take another t-norm and apply it on our previous data set in Table 6.

Judge	a	b	$\Delta_m(a,b)$
1	0.40	0.70	0.40
2	0.30	0.80	0.30
3	0.90	0.30	0.30
4	0.70	0.40	0.40
5	0.20	0.30	0.20

6	0.80	0.40	0.40
7	0.70	0.50	0.50
θ=collective outcome	0.57	0.49	0.36

Table 6: profile draw: min operator

Along with different data sets we have tried different t-norms to see if the foundation laid still holds.

We used the same data set and one more t-norm to verify our results. First we used $\Delta_m\ (x\ ,y)$, with same data set used and found results in Table 6. Taking the distances from the four consistent data sets we got Prototype $\{TFF\}$ as our solution. Now we used the data in Table 5 to calculate distance using endpoint method, the minimum distance from the majority the output was $\{TFF\}$.

For this data set my solution for all the methods come out to be same. However this is not always the case, as we checked for this using algorithm and calculating distance of some random data sets and finding solutions which were different for some data sets.

Now this is an intriguing part of this study, which is that, as initially it was only Prototype that gave importance to each individual judgment on each proposition. However when these judgments were assigned degree of membership even the endpoint satisfy supermajority responsiveness.

While we take the average to find majority output, each of the judgment is given equal weightage and thus makes it a better representation of the majority output as compared to the same approach in crisp logic, where a false gave zero weightage to that judgment. Another conclusion we draw from this is that in fuzzy judgment aggregation each of the judges and their judgments have a key role to play in deciding the conclusion and hence this is a more reliable way to come up with the best compromise between judges.

The authenticity of our method could be re-verified by applying the same methods on a different data set to get rid of the slightest chance that different data set would result in more distance for the two methods. For this we apply Łukasiewicz to another data set and analyzed the results further.

Judge	a	B	$a\Delta_l b = max\{0, a+b-1\}$
1	0.70	0.60	0.30
2	0.85	0.9	0.75
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	0.20	0.95	0.15
6	0.50	0.65	0.15
7	0.45	0.86	0.31
$\theta = collective\ outcome$	0.60	0.77	0.37

Table 7: profile draw 2: Łukasiewicz operator

Applying Łukasiewicz we calculate distance using prototype approach and take the aggregates of the distances from all the consistent judgment set and found minimum distances and solution rule as $Prototype_{fd}$={TTT}. When we applied to the same data set on endpoint we found the same solution for $Endpoint_{fd}$ ={TTT}.

In conclusion, we can say that fuzzy setting give us the best possible compromise with maximum agreement on a particular solution. This is irrespective of which of the two method you are using and this is better than the crisp approach. These two methods can be applied in different situations, and we expect different answers.

If both methods give same solution, it restricts useful of the method and that's not the goal. This will hamper the research of this thesis. Thus it's a useful insight that we found that the two distance methods give different answers when applied on the same data set.

However, the application of the min t-norm is a suggestion that it's a relatively relaxed t-norm and can be used in a situation where we are more lenient in terms of our decision in favor of the affected party. This way we can assign different t-norm based on the situation at hand and get results. This is important as we are now able to distinguish between different situations which was not present previously.

Our approach not only gives the better compromise but also is adaptable based on the situation and the kind of audience we are expecting. In a situation where the parliament has to reach a decision but due to opposition the problem remains unsolved, t-norm can help to avoid the situation.

Type of t-norm used and the kind of distance method applied varies depending on the situation, this is an important insight as this liberates us from many hunches previously faced in the field of judgment aggregation.

4.3 Judgment Revision

After this we study the other two methods that were discussed in Miller and Osherson (2009), full and output also known as judgment revision. The idea of judgment revision also stems from the 'Deliberative democratic theory', which was inspired by Elster (1998) about the ideal form of decision making as a process of conversation and actions based on supporting reasons. This way they tried to reach a consensus which was result of the continuous evolvement of the discussion people had.

This thesis sheds insightful light on the idea based on the basic deliberative democracy ideals. In the past literature they have failed to bring about a decision that is followed by both the individuals as well by the collective majority, which emphasize on the need of deliberation. This way the method can expand the aggregation procedure by 'revealing private information', which happens when the process of judgment revision takes place. Under some consideration, judges when asked to revise their opinion will share their judgment with the other colleagues, based on the discussion that takes place within the closed group the judge then decides to revise it opinion. Now this exchange of private knowledge allows for the process of deliberation and evolution of political decision based on discussion and consensus.

In fuzzy setting, all of the judgments are consistent and follow the decision rule, thus we should have a more solid reason in order to request the judges to revise their opinion. Basic idea here is to evolve judges to mold their opinion with the help of more information that they share while the discussion session takes place. Hence we look at the distance between individual and collective outcome, rather than consistent and inconsistent profile to arrive at a conclusion.

The whole process starts with the fact that distance between the majority outcome is found using Hamming distance, the judgment set that is found to have the maximum distance is then approach in an attempt to ask the judge to revise their opinion. The process then may involve presenting the judge the facts they might be missing or just the sharing of information that is either private or the view point any judge hold based on his pure subjective judgment.

The base of our research is distance, thus judgment revision in fuzzy logic becomes extracting a majority outcome that has minimum distance from each of the judgment set. The basic problem then is to minimize $\sum_i H\ (\theta, p_i)$, where θ is collective judgment set, and p_i is individual judgment set.

Judges	a	b	$\Delta_l = max\{0, a+b-1\}$
1	0.70	0.60	0.30
2	0.85	0.90	0.75
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	0.20	0.95	0.15
$\theta = collective\ outcome$	θ_1=0.65	$\theta_2 = 0.77$	θ_3=0.42

Table 8: Profile draw 3; Łukasiewicz implication

We take distance of each judgment set from the majority output θ and then improvise by reducing the aggregate distance. The distance of each of the judgment set is calculated using the Hamming distance.

$$H(\theta, \mathrm{p_i}) = |\theta - p_i|$$

When the distance between the individual judgment set and majority is calculated and then followed by taking the sum of it, giving the following calculation:

$$H(\theta, \mathrm{p_1}) = |0.65 - 0.70| + |0.77 - 0.60| + |0.42 - 0.30| = 0.34$$

$$H(\theta, \mathrm{p_2}) = |0.65 - 0.85| + |0.77 - 0.90| + |0.42 - 0.75| = 0.66$$

$$H(\theta, \mathrm{p_3}) = |0.65 - 0.70| + |0.77 - 0.70| + |0.42 - 0.40| = 0.14$$

$$H(\theta, \mathrm{p_4}) = |0.65\text{-}0.80|+|0.77 - 0.70| + |0.42 - 0.50| = 0.30$$

$$H(\theta, \mathrm{p_5}) = |0.65\text{-}0.20|+|0.77 - 0.95| + |0.42 - 0.15| = 0.90$$

$$\sum_{i=1}^{5} H(\theta, \mathrm{p_i}) = 2.34$$

Looking at the distances we find that H $(\theta, \mathrm{p_5})$ =0.9, that is fifth judge has the highest distances from the majority output. Here we ask the fifth judge to revise her opinion. We further imply that the change in decision revised should be less than 0.9, not because in this situation the distance of the 5th judge is 0.9.

We put a restriction by limiting the change in judgment to be less than 0.9 degree of membership. This process of revising an opinion as suggested earlier was a process of consideration by the judge as to what aspect he think he is missing while making the decision. The change could be because of several reason, either the judge reconsider his decision based on any facts provided to him additionally, or this could be of possible discussion among the panel of judges.

Here they discuss their own personal judgment revealing their subjective preferences and the reasoning behind it. That what is the actual democratic process should be like, the conclusion should reach after considerable discussion among the judges explaining their stance and the reasons behind it.

This information sharing may also smooth the difference in the expertise of the judges, those who have more knowledge in specific area may be able to help the entire group and reach a consensus that is better understanding of the issue at hand.

However this may be influenced by the idea that many judges who have better convincing skills are able to manipulate the entire group to vote in favor of their preferred alternative, this manipulation can be reduced when we take in account fuzzy theory where the simple yes or no is not the idea rather the degree of membership assigned, when judges assign degree of membership it still becomes difficult for judges to manipulate the whole decision in any specific direction.

The process then takes a form of an algorithm where we keep finding the judgment set that has the maximum distance and then follow the process of judgment revision with the help of their surroundings. After the fifth judge changes his opinion we find a new form of the data with new average and reduced distance of each of the judgment set from the majority.

	a	b	$\Delta_{l=}$max(0,a+b-1)
1	0.70	0.60	0.30
2	0.85	0.90	0.75
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	**0.45**	**0.95**	**0.40**
θ	**0.70**	0.77	**0.47**
		$H(\theta, 1)$	0.34
		$H(\theta, 2)$	0.56

		$H(\theta, 3)$	0.14
		$H(\theta, 4)$	0.20
		$H(\theta, 5)$	0.50
		Sum	1.74

Table 9: algorithm

After the revision, we take the aggregate distance which comes out to be

$$\sum_{i=1}^{5} H(\theta, p_i) = 1.74$$

The change in judgment was less than 0.9 which was the minimum distance required to change opinion in crisp logic in paper of Miller and Osherson (2009).

We continue these steps by taking the max distance judgment set revised until changed judgment sets are less than or equal to 2/5 of the panel of judges, we have kept this as our cut point where the process of judgment revision will stop in order to appreciate the originality of the opinions of the judges and not revise all of them. After changing Judge 5 judgment set, we find judge 2 to be at a maximum distance from majority output, revising his judgment we get

$$\sum_{i=1}^{5} H(\theta, p_i) = 1.38$$ (See Appendix III Table 3).

Further solving for minimum distance we alter judgment set 5 and 2 respectively getting a minimum of distance of 1.38. See Appendix III Table 3 and calculation of Hamming in it. Finally after altering judgment set 2 we arrive at our set cut point and reach a minimum of distance 1.38, see Appendix III Table 3.

Judges	a	b	$\Delta_l = max(0, a + b - 1)$
1	0.70	0.60	0.30
2	0.85	0.75	0.60
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	0.45	0.95	0.40

θ= collective outcome	0.70	0.74	0.44

Table 10: profile draw 3; algorithm

Sum= 1.38, Now the aggregate distance is 1.38 which is very close to the distance required in binary logic to change one judgment set.

We minimized the distance between majority outcome and individual. Also the aggregate distance we reach here is approximately equal to that in crisp logic which was bound to change if only one judge has to change its opinion. This definitely place us in a better situation than in the binary evaluation. The aggregate distance equal to just the distance between two judgment set is one milestone we have achieved using our fuzzy set theory application in judgment aggregation. This will paved ways onto the effectiveness of the collective solution and reliability of the judges on the outcome they have achieved.

After altering the judgment set we calculate the distances from the established crisp logic consistent judgment set, giving us a solution that's the better representation of the best compromise between individuals

$$Judgment\ revision_f\ = \{FTF\}$$

This means that now the output we have reached with this majority outcome is a better representation of the opinion of the judges as its closest to the majority output. After adjustment the aggregate distance is reduced hence the solution is closer and better illustration of the opinion of the judges.

This process involves not only realistic approach in decision making process, but it also caters to appreciate the basics of democratic credentials.

In political decision making process the debate and discussion only yields best interest of the individuals of the society as each representative knows the impact on its area the most and thus can either support or prevent any policy implementation keeping in view its impact. Similarly the judgment aggregation in form of agents can help to overcome the problem of decision making that has some blind fold areas that other judge might not be even informed about. Judgment revision takes all of such factors in consideration and thus the best approach to reach decision in groups.

Another factor this dissertation cater is the reduced distance a judge may require to alter his opinion, initially to change opinion judge had to completely revise its opinion from 0 to 1 or from 1 to 0. In fuzzy setting the distance is minimized to change the judgment set, meaning a less disagreement a judge may have to revise its opinion, as it will not be completely against what he thought initially.

5. Incomplete Judgment Sets:

Dokow and Holzman (2010) in their paper studies the problem of missing judgments by working on a new framework, they allowed abstentions on several issues. Dokow and Holzman (2010) further explains their model by stating that those numbers that are part of set (0-1) have to be equal to the number of issues. The problem arises when number of issues are not equal to the number of judgments and we have a set that is incomplete.

Dokow and Holzman (2010) extended the previous model presented by Wilson (1951). It was further expanded by Rubinstein and Fishburn (2006). However addition to the previous literature they allowed abstaining on judgments for any specific reason. Model was constructed in a way that abstentions are captured by a vector that has 0, 1 and another object * to be characterized by abstaining from judgment. There are two variants, one that allows agents to abstain and is determined by X*, and another variant where individuals are not allowed to deviate and thus are determined by X only. It is important to know that in both the aggregators fulfills Arrow's analogue and its axiomatic approach that are related to social welfare functions strictly.

One of them is Independence, which means the stand of society on an issue should be dependent only on individual's decision on that issue. Second property they considered was Pareto optimality, which means that society accepts any 0 or 1 position. Hence the result of Dokow and Holzman (2010) implied the theorem for linear preferences only. This work was motivated by the work of Gardenfors (2006), who stated that the condition of completeness is not realistic and hence it's not possible to have judgments that are only consistent and complete.

However Benarma et al (2010) worked in the same area by allowing judges to abstain or give neutral judgments , as well as they included an additional point by allowing judges to give their agreement or disagreement on the decision rule where they can also if not satisfy illustrate their own decision rule.

The people can express their opinion in binary form 1 or 0. In this paper they explained the idea through an example of a research funding agency who have to decide on the propositions leading to funding for an agent to his research.

The decision rule also state if all the propositions are true then they will be selected. This paper introduced the idea of judgment status, by accepting the fact that it's not always necessary for the individuals to state their belief on all the propositions at hand, they allowed their judges to give their opinion in form of binary values like 1 and 0, or they can give a neutral opinion or just abstain from giving any judgment.

They also suggested that the group decision should be reflective of their opinion on the decision rule, hence it was allowed that the judges express their opinion on all the propositions but also stating if they agree with the decision rule. If a judge agree to the decision rule then his judgment would be counted normally. But if they refuse to agree with the decision rule then they can either define a rule that they think is better or they can just give their opinion on the final decision clause.

By doing this they not only allowed for flexibility in expressing the opinion but also gave room to the judges to analyze the decision rule as well, and weigh it's applicability at the current issue. After allowing judges to express their opinion , the search then focuses on which judge has given opinion and which judges have abstained or stayed neutral, based on this there stance on decision is calculated but that's only if they agree to the decision rule in itself. Table 10 shows the example from the same paper and how they have worked along it. Judges vote on proposition "P, Q and R" as well as whether they accept the decision rule or not. Judges that have abstain assign an "X" and those that stay neutral assign "?" to their judgments, also give either 1 for agreement or 0 for disagreement.

Member	Acceptance	P	Q	R	D
M1	1	0	0	1	0
M2	1	1	X	1	1
M3	1	?	0	0	0
M4	0	1	X	1	?
M5	0	0	1	1	1

Table 10: Source: Benamara, Farah, Souhila Kaci, and Gabriella Pigozzi. "Individual opinions-based judgment aggregation procedures." *Modeling Decisions for Artificial Intelligence*. Springer Berlin Heidelberg, 2010. 55-66

M2 gives its opinion on two proposition only and also agrees to the decision rule, the decision is then drawn based on its decision on two proposition only. Although if they are following the decision rule, they have actually allotted 1 to the missing value in M2 judgment and thus calculated the decision. For M4 the decision is based on its judgment on the two propositions only, as she rejected the decision rule and stayed neutral on one proposition. M5 rejected the decision rule but gave its own decision rule. They calculated its decision based on the rule he suggested.

This all give us meaningful insights into the method that we are developing and what that has been done in literature and what further can be built on improving the area that have been worked on.

The work of Miller (2008) was then of more interest after this on subjectivity of the decision making, he worked in the area to illustrate how the aggregation problem looks like when judges have different views on the way premises are connected. In this paper they presented a model where there is no defined decision rule rather certain premises where the judges have to give their opinion, however along with opinion the judges also state the rule that they want to follow and then their decision is calculated based on that decision rule.

Miller (2008) allowed the judges to explain their decision rule where only some specific premises which they deem relevant appear, but alongside the judges also have to explain their opinion on the propositions they have stated to be irrelevant for the final decision. This basically aims to invoke the subjectivity of the decision maker and their decision will be based on their analysis and connection between the premises and how they think it will affect. If example study research about certain disease then it will evolve thinking about how the search will affect the medical filed, the kind of diseases it will cure and lives it can save and many other relevant concerns. Based on this the way these opinions are connected is then subjective and is different for every individual, this approach however sets new horizon in the field of judgment aggregation.

The conclusion they draw from this paper is that when they combine all the judgments and the decision rule each individual has followed if they could come up with rule that group follows, however the findings in the paper suggest that there is no rule that the judges unanimously follow, "⇔" suggest if and only if situation.

Voter	P1	P2	P3	P4	Decision rule	Q=vote
Voter 1	T	T	T	F	$(P1 \wedge P2) \Leftrightarrow Q$	T
Voter 2	F	F	T	T	$P1 \wedge P3 \wedge P4 \Leftrightarrow Q$	F
Voter 3	T	T	F	T	$P3 \wedge P4 \Leftrightarrow Q$	F
legislature	T	T	T	T	$Q \Leftrightarrow ?$	F

Table 11: source: Miller, M. (2008). Judgment aggregation and subjective decision-making. Economics and Philosophy 24, 205–231.

Table 11 shows each judge expresses his opinion on all propositions but then bases his final vote on a decision rule which is individual and does not necessarily comprises of all the premises. Voter 1 assumes that only premise 1 and 2 are important for the final decision ,on the other hand voter 2 suggest premise 1, 2 and 3 to be part of the decision rule, finally voter 3 considers only premise 3 and 4 are the one which should determine the decision. Looking at the table we analyze that this approach does not render a decision rule that is followed by the group in order to arrive at the decision. For future work, Miller (2008) suggest that other general logics along with probabilistic approach can help to achieve the missing and hindering factor in this approach.

In the present work we have focused on incomplete judgment set and the idea of working this in fuzzy set theory, we work to analyze how far our method fulfill the loop holes in the previous literature .

The setting in our model is based on the fact that we have already liberated our judges from the boundary of binary values, rather the judges state their opinion in the range between 1 and 0.

We in our methodology have given a far more realistic approach that on an offset allows the agent to assign proposition the exact amount of truthfulness that they seem is accurate.

This way we have already catered the problem of distances and aloofness of group decision from the individual decisions. We now further work to make our working into the area where the judgment set is consistent, closed but not complete. We base the idea of our inference from Gardenfors (2006), who stated that it's not possible in real situations to always get the judgment sets that are complete. We follow this stance and in fuzzy logic see to work out the fact where we do not have complete judgment set.

Research has shown that at numerous times judgment aggregation face the problem of incomplete judgment set. This situation arises when among the panel of judges; one or more judges refrain from giving an opinion on a proposition. These sorts of situations could be the result of such reason on the part of information the judge had while making the decision. The reason a judge may refrain from giving judgment on a certain proposition could be for the following reason

1. The judge does not have sufficient information to decide

2. The judge may consider a proposition irrelevant and abstain to give opinion
3. The judge may be indifferent between two alternatives

We assume that judges can either give their opinion in form of a degree of membership which will lie in [0, 1]. Judges can either stay neutral [N] or they can abstain [A] from giving an opinion.

Now the judges may give any opinion other than the degree of membership. Neutrality basically defines the idea that the judge is indifferent regarding that proposition. Multiple reasons could result in neutral opinion by the judge. Clause for the number of publications, the candidate might have enough publications but the judge is not satisfied with the journals and quality of the publications. In this situation when the judge seems to stay out of any controversy he may name his judgment as neutral.

Another of such example could be if the judge thinks that the actual subject matter should be the number of years a candidate has experience in and not the publications, hence in this case he can stay neutral.

On the other hand a judge may abstain from giving a judgment for two reason, one is that he might not consider a clause to be important enough and hence gives no judgment on it. Another could be when the judge thinks that the information he has regarding the clause is not enough and his decision may not be the rightful decision, in this case a judge may refrain from giving an opinion.

Among all the reasons, a judge abstains from giving an opinion and we may face the problem of incomplete judgment set. For this situation where judge hold a neutral opinion in our study we assign a degree of membership = 0.5, as per the preference theory.

Benamara et al (2010) focus on the aggregation of judgments which are incomplete. In this paper they assigned value of '1' for the case where the judge remain neutral or have abstained and agree with the decision rule. And assign '0' when the judge disagrees with the decision rule. However in my approach we have not allowed the judges to express their opinion on the decision rule but judges may abstain or stay neutral.

For the decision rule, we will be using Łukasiewicz t-norm and calculate the values for the decision clause. Important to notice here is that for every neutral judgment we assign a value of 0.5. The reason we base is on the preference theory as well as on the idea of utility maximization. The idea of preference theory states that the individual assigns a value of 1 if he completely prefers B over A and a value of 0 if he completely prefers A over B. In judgment aggregation which is the specific form of preference theory, if the judge is indifferent on a proposition, the degree of membership assigned is equal to 0.5.

Let's take an example of a situation when a research work is presented in front of a five member committee. Its funding is based on a certain criteria presented in form of certain proposition

a: $originality\ of\ the\ idea$

b: $applicability\ of\ the\ idea$

c: $if\ both\ (a\ and\ b)\ then\ funding\ for\ the\ project$

If a member abstain from giving a judgment, we arrive at a solution with the help of distance method. To calculate missing values, we minimize Hamming distance $H(\theta, j_i)$, where θ is the group average and j_i is the judgment set with missing value. We calculate this and finds the value that gives us minimum distance from the majority outcome.

Member	P	Q	$P\Delta Q = \max\{0, p + q - 1$
1	0.70	0.60	0.30
2	0.65	0.90	0.55
3	0.70	N	?
4	A	0.7	?
5	0.61	0.72	0.33
θ $= collective\ outcome$	0.67	0.73	?

Table 12: profile draw

Looking at the Table-12, member 3 decides to stay neutral on proposition “Q”. Based on preference theory we assign 0.5 value to all neutral judgments. For judgment set which has abstained judgments, Hamming distance $H(\theta, j_i)$ will be used.

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$$\min H(\theta, j_i)$$

$$H(\theta, j_4) = |0.67 - A| + |0.73 - 0.72|$$

$$min\ H(\theta, j_4) = 0.01 + |0.67 - A|$$

To minimize A should be equal to 0.67. Hence for our calculations we assign 0.67 to minimize the distance that is we assign majority average to the missing values. Calculating the entire table again we find the following values.

Member	P	Q	$P\Delta Q = \max\{0, p + q - 1$
1	0.70	0.60	0.30
2	0.65	0.90	0.55
3	0.7	N=0.50	?= 0.20
4	A=0.67	0.70	?= 0.37
5	0.61	0.72	0.33
θ=collective outcome	0.67	0.68	?= 0.35

Table-13: revised profile draw

Based on the calculations we have assigned the values to the missing judgment set. In some of previous literature Benamara et al (2010), they would either assign 1 or 0 to the missing value based on certain assumptions. The group judgment sets have less distance from individual judgment sets meaning the solution will be more realistic and better representation of the group member's opinion.

Second important finding is that in some previous literature they also deleted the entries that have the missing values, however that is not logically correct as each judge should be equally represented in the decision process more importantly for those for which they have given a judgment. This way we have reached to the solution that will be the improved compromise among the judges on a set of propositions and thus make them more practical.

As we fill our missing judgment sets, we can apply our distance methods to find our solution. Now that we have completed our data set based on the assumption of minimum distance, we now can find a solution based on any of the two methods explained above, either prototype or endpoint.

Our contribution is significant, we assign group average to missing value. This approach is less extreme than other where “1” or “0” were assigned. The distance is minimized between judgment set and group outcome, and hence the overall distance reduces as well.

Now after using the values generated to fill the missing information in our data set in Table-13, solution can be produced using any of the methods we described in previous sections. But more important aspect is that we now have a complete data set to use any distance method.

Thus this study enforces the confirmation of results we get through fuzzy logic. The distance we achieve with fuzzy set theory is less than that of crisp logic. Hence it’s a better approach to incorporate in decision making process.

6. Conclusion and Economic Application:

There has already been significant work done in this field but the aim of this study was to come up with method that leads to minimize the difference among a panel of judges required to make collective decision. This study presented four distance based methods in fuzzy setting and there were considerable improvements in results than those in crisp judgment aggregation.

Based on the methods, the judges were given liberty to express their opinion between two extremes, that is in [0, 1], and this is degree of membership or truthfulness of the proposition in view of the judge. This way the opinion is of more value as they represent the actual view point of the agents on the set of proposition. Degree of membership offered an insight to how much true the judge consider the proposition to be. This paved ways to subjectivity of the opinions of the individual as they now had better expression and the collective outcome reached is a better display of the mutual consent of the group agents.

This study was able to establish the fact that judgment aggregation in fuzzy setting is a better approach to arrive at group conclusions that have more participation by each member of the panel.

There have been important insights leading to improvements in decision making process. We have been able to improve a method in belief revision in fuzzy setting as well as to modify the established distance based method. We have proved that the distance among judgment set is reduced if the judgments are expressed between range of numbers rather than absolute yes or no, thus giving more flexibility and liberty to the judges.

The theory of belief merging in computer science had its flaws given the two tuple that had only two extremes as an option, fuzzy logic pass through these clinches and present a model that overcomes the problems faced previously.

The condition of independence which had been a debate over many years and in Miller and Osherson (2009), we are following, has been fulfilled in fuzzy modeling. Non-dictatorship axiom has not been difficult to apply; rather the degree of membership completely removes any chances of dictatorship or manipulation by a strong agent onto the group. The condition of unanimity is more reasonable here as now the distance between the solution and the majority outcome is minimized so that it is acceptable to say that the solution is unanimously agreed by the majority.

We studied advantages in fuzzy logic over the Boolean logic for the four distance methods already in the literature. Starting from prototype, the method not only provided with fewer distance but some very important additional properties.

In the paper Miller and Osherson (2009), the method had a tie between two solutions, fuzzy logic provided with one solution. The aggregate distance between the crisp set and the individual judgment set now smaller and thus a closer step toward the best compromise among the judges on a set of propositions. Endpoint on the other hand fulfilled the condition of supermajority responsiveness that it failed initially in crisp logic. Each judge is allotted equal weight while calculating the group average that involves better representation and each judgment plays its role in decision making process. This further enforces the involvement of Arrow's axioms and adherence to them. Belief revision allowed for the process of debate and discussion with the involvement of private information, which paved ways for new exploration in this field and is more realistic to be applied in practical situations where the decision is reached with mutual consent and considerable discussion. Incomplete judgment set and methodology to get rid of them removed the unrealistic approach of assigning 1 or 0 based on the choice of the researcher, or else completely deleting the judgment set. Further, methodology in incomplete set allowed for catering the area of neutral and abstentions of judgment, where the minimum distance from majority solved the problem faced in crisp logic.

Additionally, use of these methods will help to reach more coherent solutions in places that already have clause by clause voting and reach conclusion through majority voting output. Examples of such is Pakistan Parliament where a bill before becoming part of law goes through three readings and during the second reading of the bill each member is presented with the detailed clauses of bill and each member then votes on it. These are the constitutional process of lawmaking. Other than this small committee for any economic decision making goes through the process of clause by clause voting, the recent example is that of Ireland Assembly where committee of agriculture and labor decides on the clauses through majority voting. If we can adopt a better way to approach majority voting and distances in opinion than we might be helping in a lot of economic decision making and policy implementation area.

The type of data set used may play important role in deciding the distance method to be used or the kind of t-norm applicable. Prototype is a long tedious method that can be used for data sets that involve only small set of judges. In a situation like voting where large number of people cast their decision in form of a vote, it becomes impractical to use prototype to calculate distance. In this situation we can use Endpoint as it gives equal weightage to each decision and yet is less chaotic. Endpoint initially fulfilled two among three criteria of a solution rule but failed on supermajority responsiveness but when we derived crisp logic into membership functions we are able to retain supermajority responsiveness criteria, as now we are taking each of the judge judgment set into consideration while we find the solution. This is achieved as now majority outcome is calculated as the average of each of the judgment set and all the judgment set collectively decide the solution without any single judge acting as dictator.

The choice of a particular t-norm is also an imperative phase of the process of decision modeling. However, the selection of t-norm is highly dependent on the type of dataset and situation at hand. Along with different data sets we have tried different t-norms to see if the foundation laid still holds. We used different data sets and two t-norm to verify our results. Situations where data set has values closer to zero permit the use of min t-norm, this sort of a condition can arise in a political scenario where parties are least likely to agree on a certain issue. However state of affairs where all the values of degree of membership are closer to one, product t-norm is more helpful in reaching the better solution.

7. Future Work

There is a great level of imbalance between rich and poor in our society. Needless to say, every human being carries equal weight when it is a question of well-being. Therefore, taxes are collected from relatively richer people, which are then distributed amongst poorer in best possible way. But how this best way is defined? This is a question which has been a part of many debates on Social Welfare. The short answer there is no best way, but a better way. But who defines what a better way is? There has always been a dire need to accommodate poor people through social welfare programs. As we have progressed towards civilization, the inequality between the rich and poor have diminishing in theory, however, data shows this is untrue. One major reasons for this widening gap is insignificant use of research and technology.

As a society, we have been making subjective and irrational decisions on the most important aspects of the society, i.e the people. Social Welfare departments should be meticulous in deciding how best to use their budget and resources. Research has been one of the key contributors towards deriving decisions and has been neglected by most departments. Research primarily includes information gathering, assessing and analysing it before reaching a conclusion.

In the last century, technological advancements has improved the capability to perform robust research and eventually being able to make more informed decisions with the help of more information and big data. There are several tools and software constructed to manipulate the data, and take out important facts and figures. As the technology improves, ability to perform analyses improves. But the key here is to fully take advantage of what is available

This research propose judgment aggregation in fuzzy logic and calculated distance based methods. However this search leaves many doors open and areas to be further explored which can contribute significantly to the literature and will be more practical in terms of application in real life scenarios.

For future research, we have the option of considering fuzzy logic judgment set to calculate the distance. Comparison can be made if the distance is calculated from fuzzy judgment set rather than established crisp. As the distance we take is from established crisp logic judgment set, instead of using crisp set we can use fuzzy based judgment sets and then take the distance of the majority from these sets. It will be helpful as now there will be more room for the solution rule to suggest answer that are better representative of the collective outcome.

We may use fuzzy numbers in functional form that are practiced in Analytical Heretical Process. They are more liberal form of representing the judgments as it gives overlapping, and the numbers are a mix of the truthfulness of the premise. Many researcher has used this approach before but applying this in distance based methods in judgment aggregation will be very useful. Another important further insight in this area of search could be involvement of more distance methods, this could involve the theory of similarity. Where instead of employing distance metrics we can utilize the measure of similarity and then analyze how the results are different. This will help studies to overcome from the limitations we might have in our results. Also other distance method will give more insight into the properties of the method we have developed so far.

Other specific future work can be in the distance methods itself, we can employ real life examples in these methods and then with the help of secondary data analyze how close the judges went in reaching a consensus that was unanimously agreed by group. This could be done using a past decision, and then if judges are asked to vote on it again and instead of yes and no rather assign fuzzy number. We can then apply these methods and compare to the decision that was already taken and then investigate how decision in this case is different from the one we saw in crisp setting.

Finally I would like to add that decision making is an important filed, where majority of the times our fate in terms of division of resources or funding of the research is being carried out. This work is much important and each development should be tried and appreciated, to be able to have better decision making strategies and consequently, better decisions.

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9. Appendices

APPENDIX I

$$d(s_i, p_1) = \sqrt{|s_{1a} - p_{1a}| + |s_{1b} - p_{1b}| + |s_{1c} - p_{1c}|}$$

Where $s_{i=1}^{4} = \{ \{TTT\}, \{TFF\}, \{FTF\}, \{FFF\}\}$

$$\sum_{i=1}^{7} d(s_1, p_i) = d(s_1, p_1) + d(s_1, p_2) + d(s_1, p_3) + \cdots + d(s_1, p_7)$$

$\sum_{i=1}^{7} d(s_1, p_i) = \sqrt{|1 - 0.40| + |1 - 0.70| + |1 - 0.10| +} \sqrt{|1 - 0.30| + |1 - 0.80| + |1 - 0.10|} + \sqrt{|1 - 0.90| + |1 - 0.30| + |1 - 0.20|} + \sqrt{|1 - 0.70| + |1 - 0.40| + |1 - 0.10|} + \sqrt{|1 - 0.20| + |1 - 0.30| + |1 - 0|} + \sqrt{|1 - 0.80| + |1 - 0.40| + |1 - 0.20|} + \sqrt{|1 - 0.70| + |1 - 0.50| + |1 - 0.20|} = 8.06$

$\sum_{i=1}^{7} d(s_2, p_i) = \sqrt{|1-0.40| + |0.70| + |0.10|} + \sqrt{|1-0.30| + |0.80| + |0.10|}$ + $\sqrt{|1-0.90| + |0.30| + |0.20|} + \sqrt{|1-0.70| + |0.40| + |0.10|} + \sqrt{|1-0.20| + |0.30| + |0|} + \sqrt{|1-0.80| + |0.40| + |0.20|} + \sqrt{|1-0.70| + |0.50| + |0.20|}$

= 7.06

$\sum_{i=1}^{7} d(s_3, p_i) == \sqrt{|0.40| + |1-0.70| + |0.10|} + \sqrt{|0.30| + |1-0.80| + |0.10|} + \sqrt{|0.90| + |1-0.30| + |0.20|} + \sqrt{|0.70| + |1-0.40| + |0.10|} + \sqrt{|0.20| + |1-0.30| + |0|} + \sqrt{|0.80| + |1-0.40| + |0.20|} + \sqrt{|0.70| + |1-0.50| + |0.20|}$

=7.59

$\sum_{i=1}^{7} d(s_4, p_i) == \sqrt{|0.40| + |0.70| + |0.10|} + \sqrt{|0.30| + |0.80| + |0.10|} + \sqrt{|0.90| + |0.30| + |0.20|} + \sqrt{|0.70| + |0.40| + |0.10|} + \sqrt{|0.20| + |0.30| + |0|} + \sqrt{|0.80| + |0.40| + |0.20|} + \sqrt{|0.70| + |0.50| + |0.20|}$

=7.54

$Prototype_{fd}$= {TFF}

APPENDIX II

$$d(s_i, \theta) = \sqrt{|s_i - \theta_a| + |s_i - \theta_b| + |s_i - \theta_c|}$$

average of all judgment on proposition $= \theta = \{\theta_a, \theta_b, \theta_c\}$

$d(s_1, \theta) = \sqrt{|1-0.57| + |1-0.48| + |1-0.12|}$ = 1.16

$d(s_2, \theta) = \sqrt{|1-0.57| + |0.48| + |0.12|}$ = 1.02

$d(s_3, \theta) = \sqrt{|0.57| + |1-0.48| + |0.12|}$ =1.05

$d(s_4, \theta) = \sqrt{|0.57| + |0.48| + |0.12|}$ = 1.04

$$Endpoint_{fd} = s_2 = \{TFF\}$$

APPENDIX III

Belief Revision

Judges	**a**	**b**	max(0,a+b-1)
1	0.70	0.60	0.30
2	0.85	0.90	0.75
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	0.20	0.95	0.15
ϕ	0.65	0.77	0.42

H(ϕ,1)	0.34
H(ϕ,2)	0.66
H(ϕ,3)	0.14
H(ϕ,4)	0.30

H(ϕ,5)	0.9
Sum	2.34

Table 1

	a	b	max(0,a+b-1)
1	0.7	0.6	0.3
2	0.85	0.9	0.75
3	0.7	0.7	0.4
4	0.8	0.7	0.5
5	0.45	0.95	0.4
ϕ	0.70	0.77	0.47
		H(ϕ,1)	0.34
		H(ϕ,2)	0.26
		H(ϕ,3)	0.14
		H(ϕ,4)	0.20
		H(ϕ,5)	0.50
		Sum	1.44

Table 2

3)

	a	b	max(0,a+b-1)
1	0.70	0.60	0.30
2	0.85	0.75	0.60
3	0.70	0.70	0.40
4	0.80	0.70	0.50
5	0.45	0.95	0.40

ϕ	0.70	0.74	0.44
		H(ϕ,1)	0.3
		H(ϕ,2)	0.32
		H(ϕ,3)	0.08
		H(ϕ,4)	0.2
		H(ϕ,5)	0.11
		Sum	1.01

Table 3